S-2016

## Section 6.1 Discrete and Continuous Random Variables

**1. Random Variables.** Consider tossing a fair coin 3 times. The sample space would be:

S = { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }   $n = 8$

Let **X** represent the number of heads obtained. We can depict this situation in a **probability distribution of X**:

| Value | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability | 1/8 | 1/8 | 1/8 | 1/8 |

We can use the probability distribution to answer questions about the variable X such as what is $P(X \geq 1)$?

$P(x \geq 1) = P(1) + P(2) + P(3) = 7/8$ OR
$= 1 - P(0) = 1 - 1/8 = 7/8$

**Definition:** A **random variable** takes numerical values that describe the outcomes of some chance process. The **probability distribution** of a random variable gives its possible values and their probabilities.

**2. Discrete Random Variables**

**Definition:** A **discrete random variable** X takes on a fixed set of possible values with gaps between. The probability distribution of a discrete random variable X lists the values $x_i$ and their probabilities $p_i$:

| | | | | |
|---|---|---|---|---|
| **Value:** | $x_1$ | $x_2$ | $x_3$ | . . . |
| **Probability:** | $p_1$ | $p_2$ | $p_3$ | . . . |

INTEGER VALUES
USUALLY THE RESULT OF COUNTING

ex # OF PEOPLE; # OF CARS; etc.

The probabilities $p_i$ must satisfy two requirements:

1. Every probability $p_i$ is a number between 0 and 1.
2. The sum of the probabilities is 1.

To find the probability of any event, add the probabilities $p_i$ of the particular values of $x_i$ that make up that event.

**Example** - In 2010, there were 1319 games played in the National Hockey League's regular season. Imagine selecting one of these games at random and then randomly selecting one of the two teams that played in the game. Define the random variable X = number of goals scored by a randomly selected team in a randomly selected game. The table below gives the probability distribution of X:

| Goals | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Probability | 0.061 | 0.154 | 0.228 | 0.229 | 0.173 | 0.094 | 0.041 | 0.015 | 0.004 | 0.001 |

(a) Show that the probability distribution for X is legitimate.

① $0 \leq P_i \leq 1$ ✓   ② $\sum P_i = 1$ ✓

(b) What is the probability that the number of goals scored by a randomly selected team in a randomly selected game is at least 6?

$P(x \geq 6) = P(6) + P(7) + P(8) + P(9) = 0.061$

MEANING: TEAM SCORED 6 OR MORE GOALS ABOUT 6% OF TIME.

**Check Your Understanding** - Complete CYU on p. 350.

NCSU GRADES 0 1 2 3 4

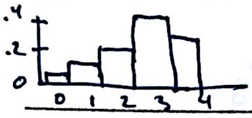
0.02 0.10 0.20 0.42 0.26

1. $P(X \geq 3) \equiv$ PROB OF A OR B.

   $P(X \geq 3) = 0.42 + 0.26 = \boxed{0.68}$

2. $P(\text{WORSE THAN C}) = P(X \leq 1)$ OR $P(X < 2)$

   $= 0.02 + 0.10 = \boxed{0.12}$

3. 


UNIMODAL, SKEWED LEFT, CENTER ≈ 2.8, NO O/L'S

**3. The Mean (Expected Value) of a Discrete Random Variable**

When analyzing shapes of distributions we used **SOCS**. If we want to know the center of a distribution of a discrete random variable we are going to have to compute the mean. The mean of a discrete random variable X is denoted by $\mu_x$. It is an average of all possible values of the random variable X but we have to take into account how many times we *expect* the values to occur. For this reason the mean is also referred to as the **expected value** of the random variable. ALSO $E(X)$

**Example:** Given the probability distribution of the discrete random variable X, find the expected value of X.

| Value | 1 | 2 | 3 |
|---|---|---|---|
| Probability | 0.5 | 0.2 | 0.3 |

$E(X) = \mu_x = (1)(0.5) + 2(0.2) + 3(0.3)$

$= 0.5 + 0.4 + 0.9$

$= 1.8$ (DISCUSS)

**Definition:** Suppose that X is a discrete random variable whose probability distribution is

**Value:** $x_1$ $x_2$ $x_3$ . . .
**Probability:** $p_1$ $p_2$ $p_3$ . . .

To find the **mean (expected value)** of X, multiply each possible value by its probability then add all the products:

$$\mu_x = E(X) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n = \sum x_i p_i$$

FORMULA SHEET

**Example:** Find the expected value of the random variable X in the NHL example and interpret the value in context. MEAN # OF GOALS ≈ 2.851 G/GAME. IF WE REPEAT RANDOM SELECTION PROCESS OVER + OVER, WE WOULD EXPECT 2.851 IN LONG RUN.

| Goals | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Probability | 0.061 | 0.154 | 0.228 | 0.229 | 0.173 | 0.094 | 0.041 | 0.015 | 0.004 | 0.001 |

$E(X) = \mu_x = (0)(0.061) + (1)(0.154) + 2(0.228) + 3(0.229) + 4(0.173) +$
$5(0.094) + 6(0.041) + 7(0.015) + 8(0.004) + 9(0.001)$

$= 2.851$

**Note:** *A common error on the AP Exam is that students incorrectly believe that the expected value of a random variable must be equal to one of the possible values of the variable. This is not the case.*

**4. The Standard Deviation (and Variance) of a Discrete Random Variable**

In order to describe the *spread* of the distribution of a discrete random variable, we are going to use the standard deviation. In order to find the standard deviation, we first compute the variance and then find its square root. The variance is the *average of the squared deviation of the possible X values from the mean*. Again, however, we must take into account how often we expect the different values of X to occur.

**Definition:** Suppose that X is a discrete random variable whose probability distribution is

| | | | | |
|---|---|---|---|---|
| **Value:** | $x_1$ | $x_2$ | $x_3$ | . . . |
| **Probability:** | $p_1$ | $p_2$ | $p_3$ | . . . |

ON FORMULA SHEET

and that $\mu_x$ is the mean of X. **The variance of X** is

$$VAR(x) = \sigma_x^2 = (x_i - \mu_x)^2 p_1 + (x_2 - \mu_x)^2 p_2 + \cdots = \sum (x_i - \mu_x)^2 p_i$$

**The standard deviation** of X, $\sigma_x$ is the square root of the variance.

**Example.** Compute and interpret the standard deviation of the random variable X in the NHL example and interpret its meaning in context. $\mu_x = 2.851$

| Goals | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Probability | 0.061 | 0.154 | 0.228 | 0.229 | 0.173 | 0.094 | 0.041 | 0.015 | 0.004 | 0.001 |

$$\sigma_x^2 = (0-2.851)^2(0.061) + (1-2.851)^2(0.154) + \ldots + (9-2.851)^2(0.001)$$
$$= 2.66$$
$$\sigma_x = \sqrt{2.66} = 1.63$$

ON AVG, A RANDOMLY SELECTED TM'S # OF GOALS IN A RANDOMLY SELECTED GAME WILL DIFFER FROM THE MEAN BY 1.63 GOALS

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**Check Your Understanding** - Complete CYU on p. 355.

| CARS SOLD | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| PROB | 0.3 | 0.4 | 0.2 | 0.1 |

① $\mu_x = 0(0.3) + 1(0.4) + 2(0.2) + 3(0.1)$
$= 1.1$ CARS $\Rightarrow$ THE L.R. AVG OVER MANY FRIDAY MORNINGS IS ABOUT 1.1 CARS SOLD.

② $\sigma_x^2 = (0-1.1)^2(0.3) + (1-1.1)^2(0.4) + (2-1.1)^2(0.2) + (3-1.1)^2(0.1) = 0.89$

$\sigma_x = \sqrt{.89} = 0.943 \Rightarrow$ ON AVG, THE # OF CARS SHOULD ON RANDOMLY SELECTED FRI. MORNINGS WILL DIFFER FROM MEAN OF 1.1 BY ABOUT 0.943 CARS.

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**5. Continuous Random Variables**

**Definition:** A **continuous random variable** X takes all values in an interval of numbers. The probability distribution of X is described by a *density curve*. The probability of any event is the area under the density curve and above the values of X that make up that event.

* REAL #'s

ex WT, AGE, TIME, etc.

* USUALLY FROM MEASURING

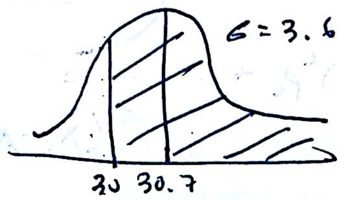
The most familiar continuous probability distribution is the (vaunted) **Normal distribution.**

**Example:** The weights of three-year-old females closely follow a *Normal distribution* with a mean of $\mu$ = 30.7 pounds and a standard deviation of 3.6 pounds. Randomly choose one three-year-old female and call her weight X. Find the probability that the randomly selected female weighs at least 30 pounds.

**State:** WHAT IS THE PROB. THAT A RANDOMLY SELECTED A FEMALE 3-YO WEIGHS AT LEAST 30 LBS?

**Plan:** LET X REPRESENT THE WT OF 3-YO FEMALES. X IS N(30.7, 3.6). WE WANT TO FIND $P(X \geq 30)$

**Do:**




CALC: 0.5770

**Conclude:**

THERE IS ABOUT A 58% CHANCE THAT THE RANDOMLY SELECTED 3-YO FEMALE WILL WEIGH AT LEAST 30 LBS.

HW: 1, 5, 7, 9, 13, 14, 18, 19, 24, 33*, 34*

P. 359.